ON XIAN-JIN LI'S CRITERION

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ABSTRACT. It seems that in order to satisfy Xian-Jin Li's criterion, the arguments of the zeta zeros must be distributed very nicely, as what happens to the Bessel function $J_\nu(z)/z^\nu$ on the real line. But the asymptotic behavior of $\xi(s)$ and numerical evidence do not support this. In this work we present a piece of numerical evidence against the Riemann hypothesis.

1. Introduction

Let $\zeta(s)$ be the analytic continuation of the function defined by [1]

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \sigma>1, \tag{1.1}$$

where $s=\sigma+it,\ \sigma,t\in\mathbb{R}$ and the Euler Γ function, [1]

$$\Gamma(z)=\sum_{n\ge0}^{\infty}\frac{(-1)^n}{n!(z+n)}+\int_1^\infty e^{-t}t^{z-1}dt. \tag{1.2}$$

Then Riemann ξ function, [1]

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1.3}$$

is an entire function satisfying and it satisfies [1]

$$2\xi(z)=1+z(z-1)\int_1^\infty\left(x^{z/2}+x^{(1-z)/2}\right)\omega(x)\frac{dx}{x}, \tag{1.4}$$

where $\omega(x)=\sum_{n\ge1}e^{-n^2\pi x}$ and

$$\xi(s)=\xi(1-s). \tag{1.5}$$

It is well-known that it has infinitely many zeros inside the critical strip $0<\sigma<1$. The celebrated Riemann hypothesis states that all the zeros are on $\sigma=1/2$.

Through years many necessary and sufficient conditions for the Riemann hypothesis have been found, and one of them is Xian-Jin Li's criterion. Let [2, 4]

$$\lambda_n=\frac{1}{(n-1)!}\frac{d^n}{ds^n}\left(s^{n-1}\log\xi(s)\right)\Big|_{s=1},\quad n\in\mathbb{N}, \tag{1.6}$$

then Xian-Jin Li's criterion can be stated as that the Riemann hypothesis is completely equivalent to the statement that $\lambda_n>0$ for all $n\in\mathbb{N}$.

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Xian-Jian Li's criterion implies that the Riemann hypothesis is valid if only if the arguments of the zeta zeros are nicely distributed. These happens to Bessel function $J_\nu(z)z^{-\nu}$, its zeros are very close the zeros of trigonometric functions on the real line. But the numerical evidence and asymptotic behavior of $\xi(s)$ do not support this kind behavior. In the following we present a numerical argument seems to disproof the Riemann hypothesis.

Here are the main ideas of our approach. At the first step we notice that the even entire function $\Xi(z) = \xi(1/2 + iz)$ is of genus 1 and it takes real values for z real. Then we define an genus zero function $f(z) = \Xi(i\sqrt{z}),\ 0 \leq \arg z < 2\pi$ via its infinite product expansion. It is clear that if we assume the Riemann hypothesis then it has only negative zeros. Since we know that given an entire function $f(z)$ of genus zero, if it has only negative zeros, then its logarithmic derivative is completely monotonic. The we write the logarithmic derivative of $f(z)$ as a Laplace transform of a positive Borel measure on $(0, \infty)$. By applying the characteristic function of a normal distribution we can express an integral related to this Laplace transform into a Fourier transform of a positive measure on the real line, and thereby we obtained a positive semidefinite quadratic form. Finally, we check numerically that the determinants of this quadratic form sometime take on negative values.

2. An Argument Against Riemann Hypothesis

It is more convenient to switch to the Riemann Xi function $\Xi(s)$. The Riemann $\Xi(s)$ function defined by

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right), \quad z \in \mathbb{C} \tag{2.1}$$

is an even entire function of genus 1, and it has an integral representation [3]

$$\Xi(z) = \int_{-\infty}^{\infty} e^{-itz}\phi(t)dt = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{(2n)!} \int_{-\infty}^{\infty} t^{2n}\phi(t)dt. \tag{2.2}$$

where

$$\phi(t) = 2\pi \sum_{n=1}^{\infty} \left\{2\pi n^4 e^{-\frac{9t}{2}} - 3n^2 e^{-\frac{5t}{2}}\right\} \exp\left(-n^2\pi e^{-2t}\right). \tag{2.3}$$

It is well known that $\phi(t)$ is positive, even and fast decreasing smooth function on $\mathbb{R}$. Clearly,

$$\Xi(it) > 0, \quad n \in \mathbb{N}_0\ t \in \mathbb{R}. \tag{2.4}$$

We observe that (2.4) and

$$\xi(\rho) = 0 \iff \Xi\left(\frac{\rho - \frac{1}{2}}{i}\right) = 0, \tag{2.5}$$

$$\rho = \frac{1}{2} + iz \iff 1 - \rho = \frac{1}{2} - iz. \tag{2.6}$$

They imply that all the zeros of $\Xi(s)$ are symmetrically located with respect to the imaginary axis, but none of them on the imaginary axis. Let

$$\rho_1,\ \rho_2, \ldots,\ \rho_n, \ldots \tag{2.7}$$

be the set of zeros of $\xi(s)$ with $\Im(\rho_n) > 0$, then

$$z_1,\ z_2, \dots,\ z_n, \dots \tag{2.8}$$

is the set of zeros with $\Re(z_n) > 0$ such that $\rho_n = \frac{1}{2} + iz_n$. The Riemann hypothesis is $\Re(\rho_n) = \frac{1}{2}, \quad n \in \mathbb{N}$ if we state it in terms of $\xi(s)$, and it is $\Im(z_n) = 0, \quad n \in \mathbb{N}$ if we state it in terms of $\Xi(z)$.

In section 2.8 of [3], Edward essentially proved

$$\frac{\xi(s)}{\xi(\frac{1}{2})} = \prod_{\rho} \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right), \quad s \in \mathbb{C} \tag{2.9}$$

with ρ and $1-\rho$ paired. In terms of $\Xi(z)$ that is

$$\frac{\Xi(z)}{\Xi(0)} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n^2}\right), \quad z \in \mathbb{C}. \tag{2.10}$$

Since $\xi(s)$ is an entire function of genus 1, then

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n^2}\right) = \sum_{n=0}^{\infty} \frac{z^n}{(2n)!} \frac{\int_{-\infty}^{\infty} t^{2n}\phi(t)dt}{\int_{-\infty}^{\infty} \phi(t)dt} \tag{2.11}$$

defines an entire function of genus 0. Furthermore,

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n^2}\right) = \frac{\Xi(i\sqrt{z})}{\Xi(0)}, \quad 0 \le \arg z < 4\pi. \tag{2.12}$$

If the Riemann hypothesis were true, then $z_n^2 > 0,\ n \in \mathbb{N}$ and $f(z) > 0$ for $z > -z_1^2$, then $\log f(z)$ is well-defined for $z > -z_1^2$ and

$$(-1)^k \partial_z^k \left(\frac{f'(z)}{f(z)}\right) = \sum_{n=1}^{\infty} \frac{k!}{(z + z_n^2)^{k+1}} > 0, \quad z \ge 0,\ k = 0, 1, \dots, \tag{2.13}$$

thus $\frac{f'(z)}{f(z)}$ is completely monotonic on $[0, \infty)$, [6, 5]. Therefore, there is a positive Borel measure $\mu(x)$ on $(0, \infty)$ such that

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{z_n^2 + z} = \int_0^{\infty} e^{-zt} d\mu(t), \quad z \ge 0. \tag{2.14}$$

Since $g(z)$ is indefinitely differentiable at $z = 0$, then $\int_0^{\infty} x^n d\mu(x) < \infty$, $n = 0, 1, \dots$.

On the other hand, by the functional equation, we have

$$\Xi(i\sqrt{z}) = \xi\left(\frac{1}{2} - \sqrt{z}\right) = \xi\left(\frac{1}{2} + \sqrt{z}\right) \tag{2.15}$$

and

$$g(z) = \frac{f'(z)}{f(z)} = \frac{\xi'\left(\frac{1}{2} + \sqrt{z}\right)}{2\sqrt{z}\xi\left(\frac{1}{2} + \sqrt{z}\right)} = -\frac{\xi'\left(\frac{1}{2} - \sqrt{z}\right)}{2\sqrt{z}\xi\left(\frac{1}{2} - \sqrt{z}\right)}. \tag{2.16}$$

Then the function

$$g(z) = \frac{\xi'\left(\frac{1}{2} + \sqrt{z}\right)}{2\sqrt{z}\xi\left(\frac{1}{2} + \sqrt{z}\right)} = -\frac{\xi'\left(\frac{1}{2} - \sqrt{z}\right)}{2\sqrt{z}\xi\left(\frac{1}{2} - \sqrt{z}\right)} \tag{2.17}$$

is completely monotonic for $z \geq 0$. Hence,

$$g(z^2) = \frac{\xi'\left(\frac{1}{2}+z\right)}{2z\xi\left(\frac{1}{2}+z\right)} = -\frac{\xi'\left(\frac{1}{2}-z\right)}{2z\xi\left(\frac{1}{2}-z\right)} = \int_0^\infty e^{-z^2 t} d\mu(t), \quad z \geq 0. \tag{2.18}$$

From the characteristic function of the normal distribution we get

$$e^{-tx^2} = \int_{-\infty}^{\infty} \frac{e^{-y^2/(4t)}}{2\sqrt{t\pi}} e^{ixy} dy, \quad t > 0, \tag{2.19}$$

then

$$g(z^2) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{izy} dy \left(\int_0^\infty e^{-y^2/(4t)} t^{-1/2} d\mu(t) \right). \tag{2.20}$$

Since

$$\int_{-\infty}^{\infty} \int_0^\infty e^{-y^2/(4t)} t^{-1/2} \frac{d\mu(t)}{2\sqrt{\pi}} dy = \int_0^\infty d\mu(t) \left\{ \int_0^\infty \frac{e^{-y^2/(4t)} dy}{2\sqrt{t\pi}} \right\} = \int_0^\infty d\mu(t) < \infty,$$

then

$$\nu(y) = \int_0^\infty e^{-y^2/(4t)} t^{-1/2} \frac{d\mu(t)}{2\sqrt{\pi}}, \quad y \in \mathbb{R} \tag{2.21}$$

defines a bounded positive measure on $\mathbb{R}$. Then we should have for all $n \in \mathbb{N}$ and $z_i \in \mathbb{R},\ c_i \in \mathbb{C},\ i = 1, \ldots, n$,

(2.22)

$$\sum_{i,j=1}^n g\left((z_i - z_j)^2\right) c_i \overline{c_j} = \sum_{i,j=1}^n \frac{\xi'\left(\frac{1}{2} + z_i - z_j\right) c_i \overline{c_j}}{2\left(z_i - z_j\right) \xi\left(\frac{1}{2} + z_i - z_j\right)} = \int_{-\infty}^{\infty} \left| \sum_{j=1}^n c_j e^{i z_j y} \right|^2 d\nu(y) \geq 0,$$

consequently,

$$\det\left(g\left((z_i - z_j)^2\right)\right)_{i,j=1}^n \geq 0, \tag{2.23}$$

for all $n \in \mathbb{N}$ and $z_i \in \mathbb{R},\ i = 1, \ldots, n$.

However, if we let $n = 3,\ z_1 = 1,\ z_2 = 2,\ z_3 = 3$ we get

$$\det\left(g\left((z_i - z_j)^2\right)\right)_{i,j=1}^3 \approx -0.00153356, \tag{2.24}$$

if $n = 4,\ z_1 = 5,\ z_2 = 6,\ z_3 = 7,\ z_4 = 8$ we get

$$\det\left(g\left((z_i - z_j)^2\right)\right)_{i,j=1}^4 \approx -0.0000695685, \tag{2.25}$$

which is impossible.

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